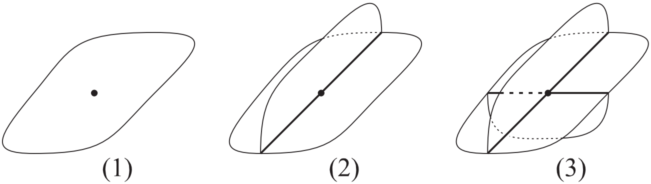
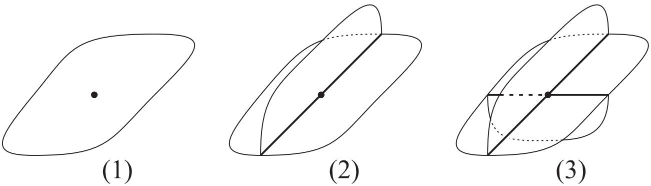
Shadow complexity of four-manifolds

Bruno Martelli

12 december 2017

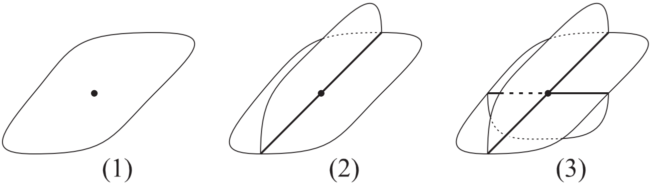


Neghbourhoods of points in a simple polyhedron P



Neghbourhoods of points in a simple polyhedron P

$P \subset \mathrm{int}(M)$ is a *spine* if $M \setminus P$ consists of:



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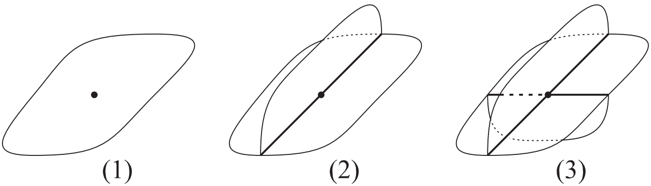

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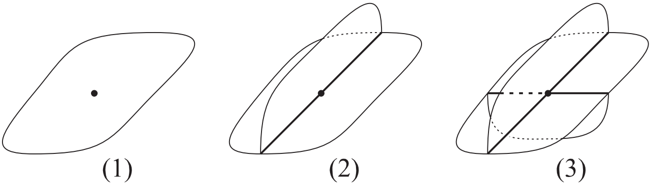

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Examples: $S^2 \subset S^3$ and $\mathbb{RP}^2 \subset \mathbb{RP}^3$.

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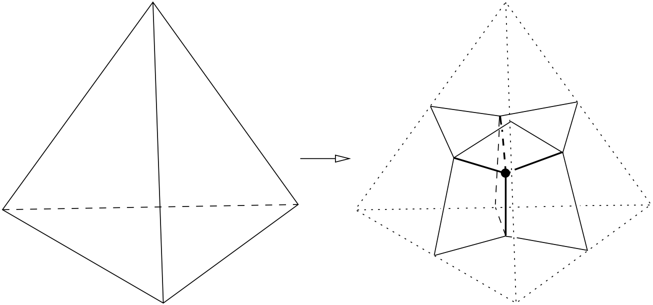

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- There are finitely many irreducible, ∂-irreducible, anannular 3-manifolds M with fixed c
- For such manifolds, if $M \neq S^3, \mathbb{RP}^3, L(3,1)$ then $c(M)$ is the minimum number of tetrahedra in a (ideal) triangulation of M

c	0	1	2	3	4	5	6	7	8	9	10	11	12
lens	3	2	3	6	10	20	36	72	136	272	528	1056	2080
other S^3	.	.	1	1	4	11	25	45	78	142	270	526	1038
$\mathbb{R}^3$	.	.	.	.	.	.	6	.	.	.	.	.	.
Nil	.	.	.	.	.	.	7	10	14	15	15	15	15
$\mathrm{SL}_2\mathbb{R}$	.	.	.	.	.	.	.	39	162	513	1416	3696	9324
Sol	.	.	.	.	.	.	.	5	9	23	39	83	149
$\mathbb{H}^2\times\mathbb{R}$	.	.	.	.	.	.	.	.	2	.	8	4	24
$\mathbb{H}^3$	.	.	.	.	.	.	.	.	.	4	25	120	459
non-geo	.	.	.	.	.	.	.	4	35	185	777	2921	10345
total	**3**	**2**	**4**	**7**	**14**	**31**	**74**	**175**	**436**	**1154**	**3078**	**8421**	**23434**

The closed irreducible orientable 3-manifolds of complexity ≤ 12.
From the atlas of 3-manifolds `http://matlas.math.csu.ru`

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For the non-orientable ones with $c \leq 11$, see *Regina* [Burton]

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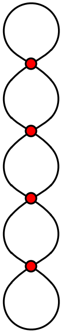

Lens spaces

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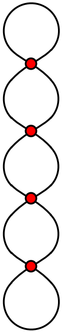

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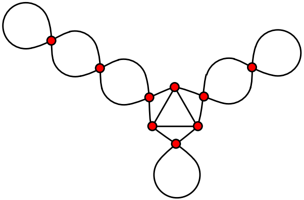

Seifert manifolds over S^2 with three singular fibres

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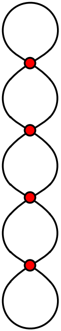

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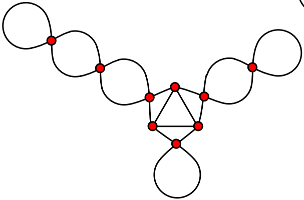

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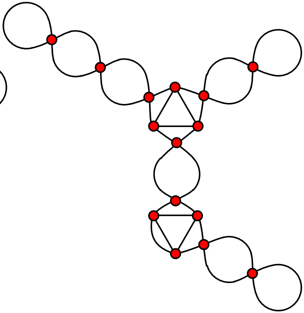

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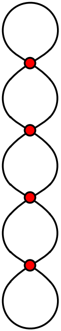

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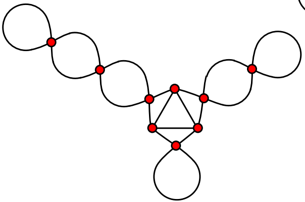

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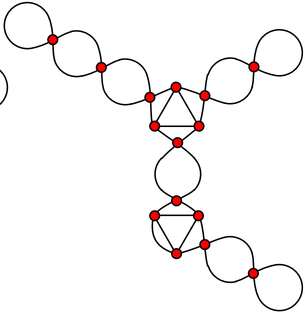
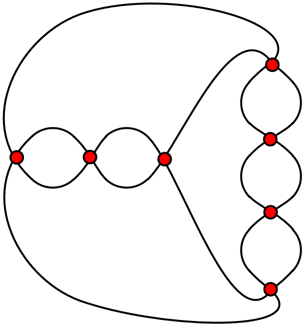


Graph manifolds

Proved for some infinite families [Jaco, Rubinstein, Tillmann 2009]

Three families of Seifert manifolds have more efficient spines:



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$$m, n \geq 1$$

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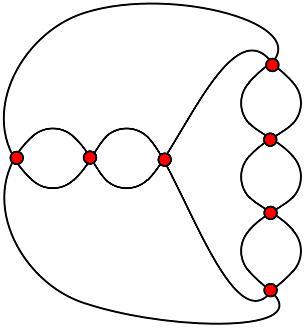

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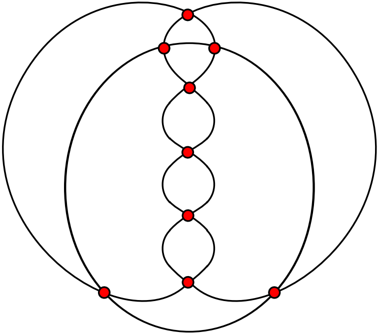

$\big(S^2,(2,-1),(3,1),(n+5,1)\big)$

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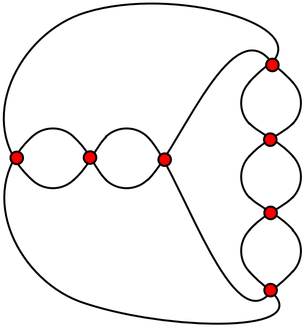

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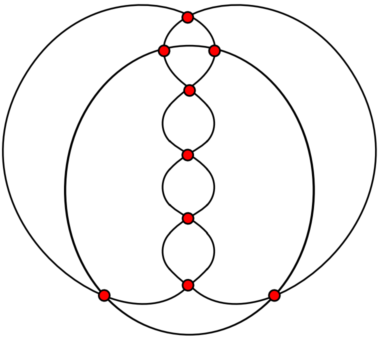

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The third family yields $\left(S^2, (2,-1), (3,1), (p,q)\right)$ with $p/q > 5$.
[Martelli, Petronio 2000].

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Triangulation

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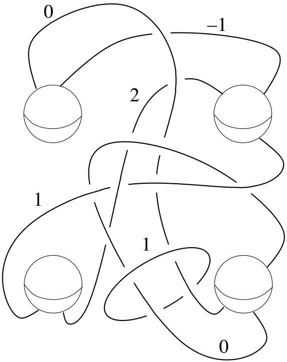

Kirby diagram

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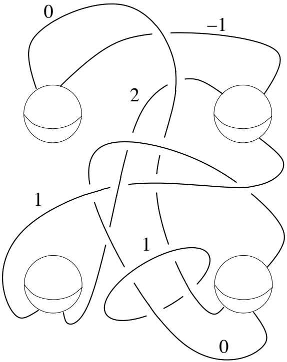

Kirby diagram

No need to draw 3- and 4-handles (Laudenbach, Poenaru 1972)

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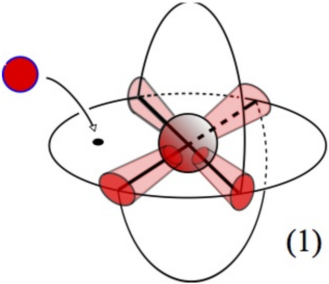
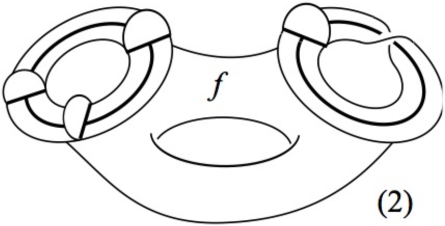


(1)



(2)

Every region f is equipped with a *gleam* in $\frac{1}{2}\mathbb{Z}$, and conversely the gleams determine M (Turaev 1994).

Every $\alpha \in H_2(M, \mathbb{Z})$ may be represented as

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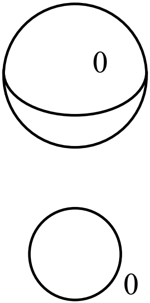
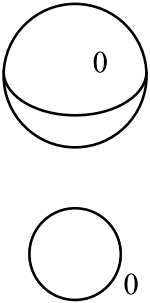
In particular, if $\Sigma \subset P$ is a surface, then

$$\Sigma \cdot \Sigma = \sum_{f \subset \Sigma} \operatorname{gleam}(f).$$



S^4



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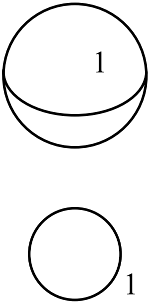
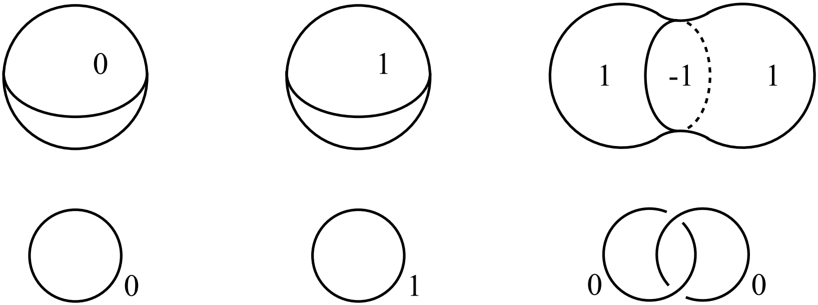
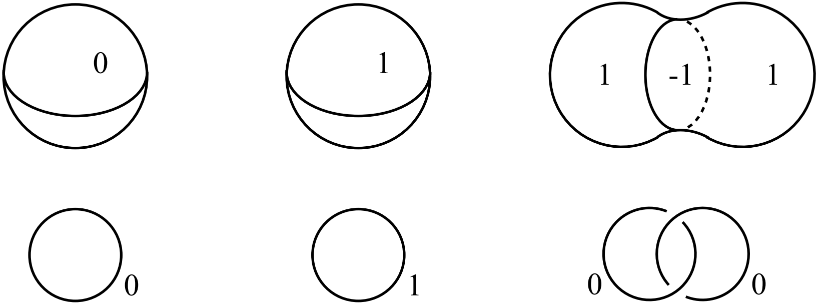


$\mathbb{CP}^2$



S^4 $\mathbb{CP}^2$ $S^2 \times S^2$



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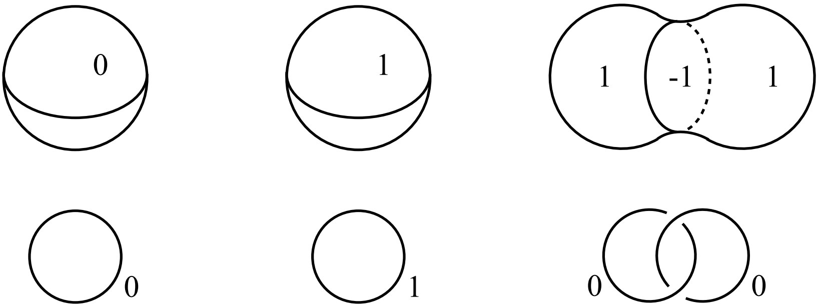
The *shadow complexity* $c(M)$ of a compact 4-manifold M is the minimum number of vertices in a simple spine of M.



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$$c(S^4) = c(\mathbb{CP}^2) = c(S^2 \times S^2) = 0.$$

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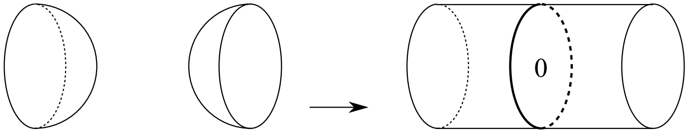

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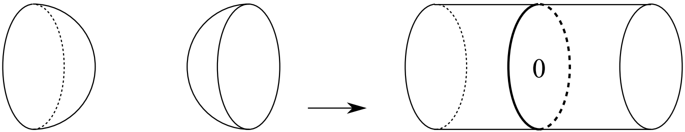

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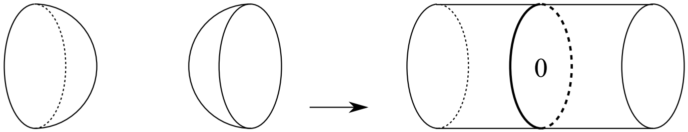

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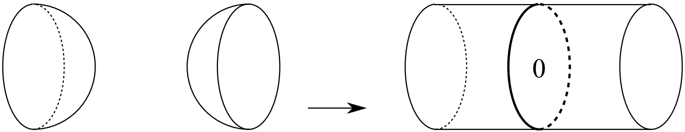

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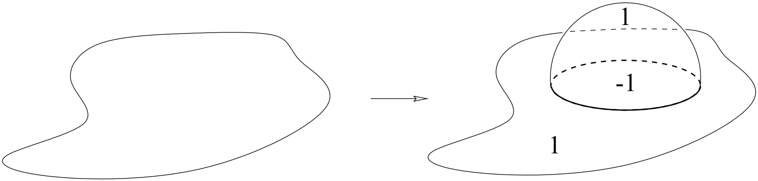

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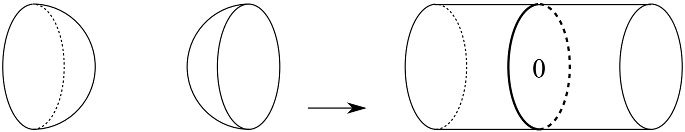

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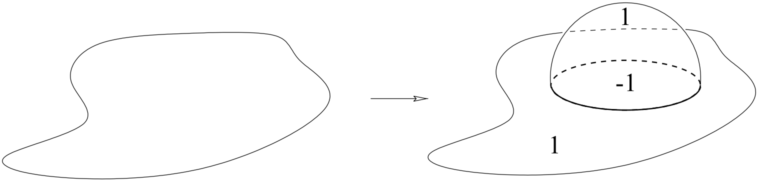

$$c(DM) \leq c(M).$$

Theorem (M. 2011)

The closed orientable smooth four-manifolds M with $c(M) = 0$ are precisely those of the type

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Corollary

The simply connected ones are:

$$S^4, \quad \#_h \mathbb{CP}^2 \#_k \overline{\mathbb{CP}}^2, \quad \#_h (S^2 \times S^2).$$

Conjecture

The closed orientable smooth four-manifolds M with $c(M) \leq 1$ are precisely those of the type

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where W is the double of a thickening of a P with $c(P) \leq 1$, or of a P containing $\mathbb{RP}^3$.

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where W *is the double of a thickening of a* P *with* $c(P) \leq 1$*, or of a* P *containing* $\mathbb{RP}^3$*.*

We have $c(\mathbb{RP}^3 \times S^1) = 1$.

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None of these four-manifolds is aspherical.

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- More elaborate simply connected manifolds. We have $c(K3) \leq 14$ [Costantino 2006].
- Aspherical manifolds.
- Manifolds of signature $h \neq 0$ that are not $M \#_h \mathbb{CP}^2$.
- Manifolds with intersection form $nE_8 \oplus mH$.

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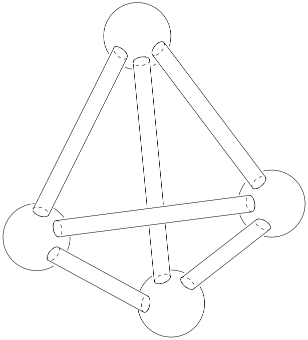
- Cusped hyperbolic manifolds at the other components of SP.

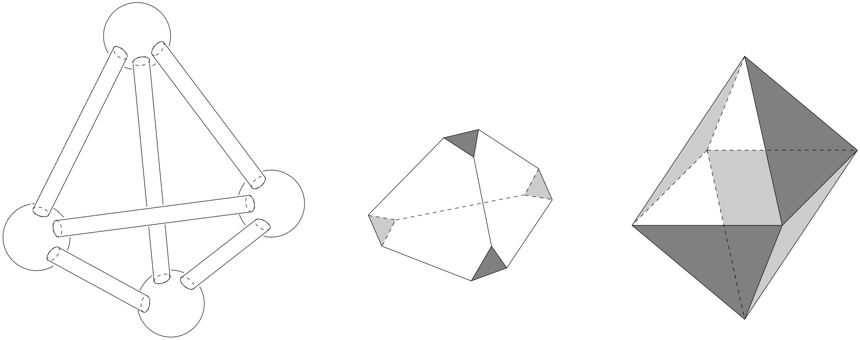
We get a *Minsky block* over every vertex of P:



Let P thicken to M. Let SP be the singular part of P. The boundary ∂M decomposes into:

- A circle bundle at every region of P.
- A pair-of-pants bundle at every circle in SP.
- Cusped hyperbolic manifolds at the other components of SP.

We get a *Minsky block* over every vertex of P:



[Costantino, D. Thurston 2008]

Goal: understand when $\partial M \cong \#_h(S^2 \times S^1)$.

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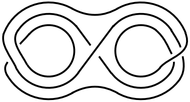
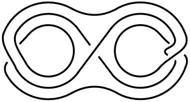
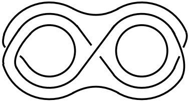


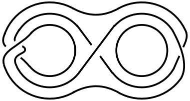
X_1



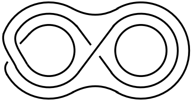
X_2



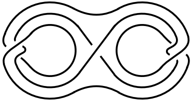
X_3



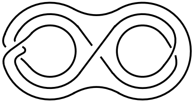
X_4



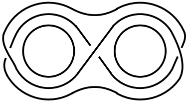
X_5



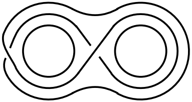
X_6



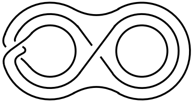
X_7



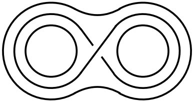
X_8



X_9

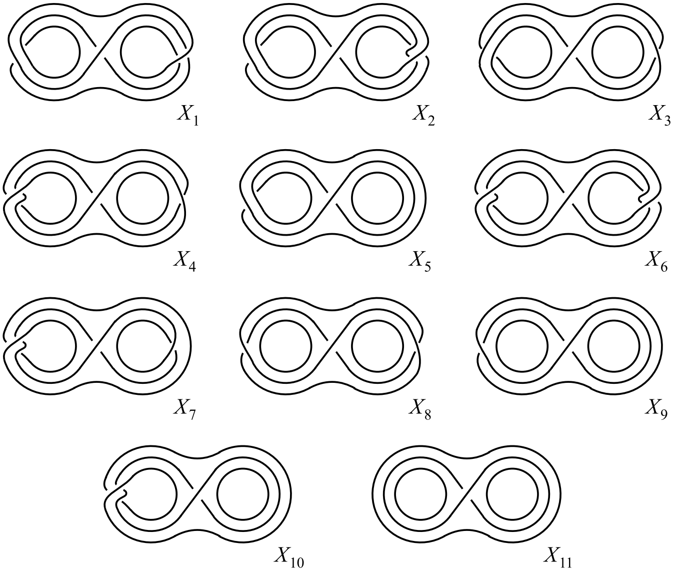


X_{10}



X_{11}

Goal: understand when $\partial M \cong \#_h(S^2 \times S^1)$.



Use SnapPy [Weeks, Culler, Dunfield]